# Assignment 1.

Basic techniques.

This assignment is due Wednesday, Jan 27. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Represent the following complex numbers in trigonometric form: (a) $1+i$, (b) $-1+i$, (c) $-1-i$, (d) $1+i\sqrt{3}$, (e) $-1+i\sqrt{3}$, (f) $\sqrt{3}-i$.

(2) Calculate
(a) $\frac{1+i\tan\alpha}{1-i\tan\alpha}$ (where $\alpha \in \mathbb{R}$), (b) $\frac{(1+i)^{2015}}{(1-i)^{2013}}$.

(3) Calculate
(a) $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$, (b) $(a+b)(a+b\omega)(a+b\omega^2)$,
where $\omega = -\frac{1}{2}+\frac{1}{2}\sqrt{3}\cdot i$. (Hint: use that $\omega^3 = 1$.)

(4) Prove the identity $|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$. (By the way, what is the geometric interpretation of this identity?)

(5) By a purely geometric argument, prove that

$$|z-1| \leq ||z|-1| + |z||\arg z|$$

(*Hint:* Draw a picture. The latter term is the length of an arc.)

(6) Draw regions on the complex $z$-plane defined by the following relations:
(a) $|z-z_1| = |z-z_2|$ (c) $\operatorname{Re} z + \operatorname{Im} z < 1$ (e) $\operatorname{Re}\frac{z-z_1}{z-z_2} = 0$
(b) $0 \leq \operatorname{Re}(iz) \leq 1$ (d) $\operatorname{Im}\frac{z-z_1}{z-z_2} = 0$

(7) Prove that any complex number of absolute value 1 (except for $z=-1$) can be represented as

$$z = \frac{1+it}{1-it},$$

where $t$ is a real number (Hint: compare to 2a.)

(8) $Az\overline{z} - E\overline{z} - \overline{E}z + D = 0$ is an equation of a circle ($E \in \mathbb{C}$, $A, D \in \mathbb{R}$, $A \neq 0$). Find its center and radius.

(9) Describe the family of curves on the complex $z$-plane with equations
(a) $\operatorname{Re}\frac{1}{z} = C$, (b) $\operatorname{Im}\frac{1}{z} = C$,
where $C$ is an arbitrary real number. (*Hint:* Use $1/z = \overline{z}/z\overline{z}$, $\operatorname{Re} w = (w+\overline{w})/2$, complex equation of a circle and problem 8 above.)

(10) Use the fact that $1+\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha = \operatorname{Re}(1+z+z^2+\cdots+z^n)$, where $z = \cos\alpha + i\sin\alpha$, to find a trigonometric expression for $1+\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha$.

(11) Use De Moivre's formula ($(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$) to express $\cos 5\varphi$ through $\sin\varphi$ and $\cos\varphi$.
Comment. If you look at your obtained expression closely, you can observe that you can eliminate $\sin\varphi$ by using $\cos^2\varphi + \sin^2\varphi = 1$, so $\cos 5\varphi$ is a *polynomial* of $\cos\varphi$. The same is true for $\cos n\varphi$. The corresponding polynomials are called *Chebyshev* polynomials and they are amazing every which way.